EIGHT HUNDRED YEAR OLD RABBIT PROBLEM BREEDS VAST MATHEMATICAL IMPLICATIONS

OR

HOW, ONE NIGHT, BETWEEN MIDNIGHT AND

2:00 AM, I SQUARED THE CIRCLE......WELL, ALMOST

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By

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EIGHT HUNDRED YEAR OLD RABBIT PROBLEM BREEDS VAST MATHEMATICAL IMPLICATIONS

Our central topic is the circle, a figure which can be constructed with great ease and appears quite simple in execution, yet this easily drawn planar object can only be classified as feminine, for she is inscrutable, full of curves and mystery.

Her circumference can scarcely be embraced with comprehension, for, extended to 100,000 decimal places, her value of pi still gives no clue to her true girth: the accurate figures obtained for pi continue apparently at random with no recognizable repetitions. In similar fashion her area, derived as it is from her perimeter, is likewise indeterminate.

Although she is elusive and permits noone to scrutinize her in naked, absolute, terms, there are certain geometrical approaches which reveal her inward secrets by pure mathematical reaction to systematic intrusions of her being. For practical purposes, she can be scanned, and used. This paper will briefly set forth some exciting relations which are peculiar attributes of this perplexing entity.

Let us begin with a problem, posed nearly eight hundred years ago by a remarkable Italian mathematician , Leonardo of Pisa, who bore the nickname Fibonacci (abbreviation for filius Bonacci), and recorded in his book Liber Abacci: "How many pairs of rabbits are born of one pair in a year?" It is assumed that every month a pair of rabbits produces another pair, and that rabbits begin to bear young two months after their own birth.

The chart simplifies the problem. Since the first pair produces issue in the first month, in this month there will be two pairs. Of these one pair, namely the first one, gives birth in the following month, so that in the second month there will be 3 pairs. Of these three pairs only 2 pairs will produce issue in the following month, thus in the third month there will be five pairs of which only 3 pairs will produce issue, thus in the fourth month there will be 8 pairs. In the twelfth month there will be 377 pairs. (A lot of inbreeding there.)

The pattern which emerges is obvious and is shown in Fig. 1. Observe that each term equals the sum of the two preceding terms. Today this is called a recurrent sequence. This particular sequence is called the Fibonacci sequence and its terms are known as Fibonacci numbers.

This innocent rabbit problem solves, in a very exacting manner, the value for Euclid's phi, or what is now variously called the Golden Mean, Golden Proportion, or Golden Rectangle. The "rabbit" sequence can be continued indefinitely, but for our purposes let us stop at the numbers 317,811 followed by 514,229. These Fibonacci numbers represent the total number of rabbits produced by the twenty-sixth and twenty-seventh months of breeding , and represent as well the power of rabbits and of the compounding of interest. Divide the larger into the smaller and the answer is 0.618033989, while the larger divided by the smaller yields the value of 1.618033989, a unique reciprocal relationship separated by exactly 1, with identical decimal fractions. The last is a very important and ubiquitous value, designated by the 21st letter of the Greek alphabet, phi, a value equipped to shed light on the shadowy and evasive nature of the circle.

Archimedes, Eratosthenes, Euclid, Plato, Pythagoras, and Thales, among others, relished the study of geometry; Pythagoras with his insightful pronouncement of the relation of squares of the sides of a right triangle, known to every schoolboy and girl. We learned it by rote and without question. Let us examine the theorem with the aid of a demonstration and a diagram. These should make the concept less awesome or forbidding. The demonstration is based on an observation by Thales.

While it does not immediately occur to one that the only right triangles which can be constructed are circle related, it was recognized over two thousand years ago. The truth of it is easily demonstrated. The hypotenuse of any right triangle is the diameter of a circle, as shown. The only right triangles which can be constructed have, as a meeting point for their sides, a position on the rim of the circle as is readily seen. See Fig. 3.

(DEMONSTRATION)

One of the most satisfying of all the diagrammed conceptual explanations of the validity of the Pythagorean Theorem is a 5 stage manipulation of the area of the hypotenuse squared into the confines of the two areas of the squares of the sides. Seeing this will take a bit of the mystery away from the theorem itself, and perhaps lift a part of the veil of mystery from the circle herself.

(ILLUSTRATION) , Fig. 2.

The explanation is straightforward. Simply remember the area rule for parallelograms, base times height, thus the parallelograms formed fit exactly into the upper squares. These moves complete the derivation of the Theorem of Pythagoras, proving its validity.

The applications of the theorem are endless and, always remember this, it is one of the revealed secrets of the circle and derives its special and quite amazing properties from the circle itself, as has been shown.

`The Golden Rectangle or Golden Mean was mentioned earlier. Euclid solved this problem, or at least set forth the solution for it. Egyptian paintings incorporated the idea of extreme and mean a thousand or more years earlier than Euclid's life span, but he is given the credit for its geometrical solution, which is a simple and elegant one. Johann Kepler, no slouch of a mathematician and geometer himself, said of the Theorem of Pythagoras that "it was like gold," but he was so smitten by Euclid's solution of the mean and extreme segmented line that he pronounced it, in comparison, "like a precious gem."

Let us first look at the problem, and then see how simply and wisely Euclid solved it.

This is the problem. Take a line, of whatever unit length, and separate it in such a way that certain ratios are established.

(ILLUSTRATION), Page 3., Notes

Note that the values are exactly the same as our earlier values derived from the relation of successive Fibonacci numbers.

Now Mr. Yost, a very practical man, after enduring the reading of my paper on Magic Squares, asked the question "What good is it? What usefulness does it have, if any?" and "Can you make any money with it?" Mr.

Yost will be pleased to know that these things do have practical applications, some of which will later be examined.

But for the time, let us briefly cover the history that led to these discoveries.

Counting began in prehistory with men scratching marks on bones using sharpened flintstones, but real enlightenment and advanced mathematical skills arose in China, India, Persia (or ancient Babylonia), and Egypt. Written records are revealing.

Geometry began in Egypt for very practical reasons. Every year the Nile would overflow and destroy markers, so the land had to be resurveyed. Derivation of the word geometry is "Geo" for land, or earth, --- "metry" for measure ---measuring the earth. The Egyptians were not concerned about advanced mathematics: they were involved strictly in solving their unique practical problems.

There is ample indication that both the Egyptians and Babylonians had a rudimentary concept of the Pythagorean Theorem long before Pythagoras committed the idea to parchment. Both knew of the 3, 4, 5 triangle, whose whole numbers formed a perfect right angle, and in the literature there was also the 5, 12, 13 triangle, another right angle triangle. Possessing whole numbers, these were easily solved.

There is even evidence that the Egyptians knew about the mean and extreme relation set forth by Euclid, from examination of wall paintings made a thousand or two years before Euclid.

The Orientals had the abacus and a well developed system in place: the Indians, in between, learned from their neighbors and proved innovative as well.

It is not surprising that the threads of mathematical knowledge came together at the land bridge of the continents and brought to the philosopher mathematicians the flowering of millennia of insights, discoveries and advancements in the ancient civilizations. Now, back to Mr. Yost's quest for usefulness. The first usefulness of phi to the geometer who works solely with compass and straight edge is the ability the value phi minus one gives him to construct a pentagon, or a star pentagram. Four ways to use the value and establish an inscribed perfect five sided figure are in your notes, on Page 4.

(ILLUSTRATIONS) Page 4., Notes

It is thus readily seen that there is a five-ness or ten-ness to the phi minus one's relation to the circle.

Further experimentation led to two additional perfect reciprocals . One is produced by the rotation of a side of an inscribed square. This cuts the radius at a distance of 0.414213562 from the circle's center (See Page 6.) Its reciprocal is 2 plus the same decimal fraction, and another was produced by the bisected balance beyond phi minus one over the sum of this same quantity plus phi minus one. This value was 0.236067978, which had a reciprocal of 4 plus the same decimal fraction. (See Page 7.) Armed with the three amazing values which all possessed perfect reciprocals, values no doubt well known to others but not before then known by me, I sought some further significance, and it occurred to me one night, between midnight and two AM that, since these values were all circle related, they might be used in the two thousand year old problem of squaring the circle, using only compass and straight edge.

In a matter of minutes I found that the sum of 0.414213562 plus twice the value 0.236067978 gave a length on the radius which, when doubled and squared was 3.142461872, showing that the side length was in error by only three parts in ten thousand. It was refreshing to learn from the book SACRED GEOMETRY that a method used for hundreds of centuries by church geometers produced a value for pi of 3.1755, more than one percent off the correct value for pi. Their method was perhaps more "sacred" than accurate. At the same time I readily admit that my own approach is sadly flawed. (See Page 7, Notes)

(ILLUSTRATION)

The more complete solution, taking pi accuracy to nine decimal points, involves either a continued scheduled bisection of the radius or of the described derived value, either of which leads to great accuracy. I will not abuse your ears with the details, which are set forth in the handout you received and constitute a part of the paper which will not be read, but will be available for any reader of the paper to examine. The final accuracy, for a circle of diameter 20 kilometers, about 12.6 miles, is within .02 millimeter, narrower than a pencil line, certainly acceptable within engineering limits.

The total expression which yields the correct value for the side of a square whose area equals that of a circle of unit radius is given at the bottom of Page 7 of the Notes.

(ILLUSTRATION)

Please note that the final expression is made up of ones and twos altogether, with the exception of the exponents assigned in the final brackets. This expression reflects the geometrical approach using only compass and straight edge.

Squaring the circle involves not only the construction of a square of equal area, but a square of equal perimeter as well. The second construction is easier, and requires only one figure for illustration. See Page 9. for the illustration and its mathematical equivalent ,which should be self explanatory.

(ILLUSTRATION)

On Page 8 of the Notes is a remarkable relationship I discovered within the past week. By extending the line of the circle's diameter and repeating, at one-half radius intervals, the earlier operations performed within the circle, an infinite number of these perfect reciprocals could be formed. No doubt it has been done before, but the discovery of it was especially pleasing to me.

Now I'm sure that I've had far more fun reading this than you have had in hearing it. So let us leave it altogether, except for a final comment or two on the usefulness of the Golden Rectangle and its applications.

First, do you recognize the Golden Rectangle in this pastel work? It is by Jo Anne Seigenthaler of Nashville. Let me explain. The proportion of width to height is very nearly that of 1 to phi, or about1.618 etc. In addition, if you look carefully you will see a square, from which a mid-point to corner line could inscribe an arc crossing the right corner of the painting. Thus phi has an application in art.

Greek architecture used phi in designing. Numerous phi relations are contained in the Parthenon. Gothic architecture drew heavily upon the relationship (See Page 10). Even the stock market analysts swear by Fibonacci. Somehow the mathematical perfection affects us by a certain pleasant resonance, a sort of optical titillation of the right brain cells, a factor which might even have some musical implications which may yet be explored, for we do not hear music as simple quantitative differences in sound wave frequencies, but rather the logarithmic, proportional differences between frequencies.

Because you may find it difficult and I may find it wounding for you to comment on what thus far has been read; out of sympathy both for you and for myself the paper will conclude on a different note.

James R. Newman asks the question, in his four volume coverage of THE WORLD OF MATHEMATICS, "Do mathematical truths reside in the external world, there to be discovered by men, or are they man-made inventions? Does mathematical reality have an existence and a validity independent of the human species or is it merely a function of the human nervous system?"

G. H. Hardy, the brilliant British mathematician said "I believe that mathematical reality lies outside us, and that our function is to discover or observe it, and that the theorems which we prove, and which we describe grandiloquently as our own 'creations' are simply our notes of our observations."

F. W. Bridgman, an outstanding physicist, disagreed. "It is the merest truism," he said, "evident at once to unsophisticated observation, that mathematics is a human invention."

James Newman and Edward Kasner agree with Bridgman, and say "We have overcome the notion that mathematical truths have an existence independent and apart from our own minds. It is even strange to us that such a notion could ever have existed."

So--what do you think? Is the math on the far side of the milky way or in other galaxies like ours, because it is innate in the nature of reality, or is our math just another of our fallible human contrivances? I await your philosophical comments with interest.

FIBONACCI'S RABBIT PROBLEM

A pair	1	Thirteenth	610
First (Month)	2	Fourteenth	987
Second	3	Fifteenth	1,597
Third	5	Sixteenth	2,584
Fourth	8	Seventeenth	4,181
Fifth	13	Eighteenth	6,765
Sixth	21	Nineteenth	10,946
Seventh	34	Twentieth	17,711
Eighth	55	Twenty-first	28,657
Ninth	89	Twenty-second	46,368
Tenth	144	Twenty-third	75,025
Eleventh	233	Twenty-fourth	121,393
Twelfth	377	Twenty-fifth	196,418
		Twenty-sixth	317,811
		Twenty-seventh	514,229

Fig. 1.

TAKE TWO NUMBERS IN SEQUENCE

317,811, 514,229

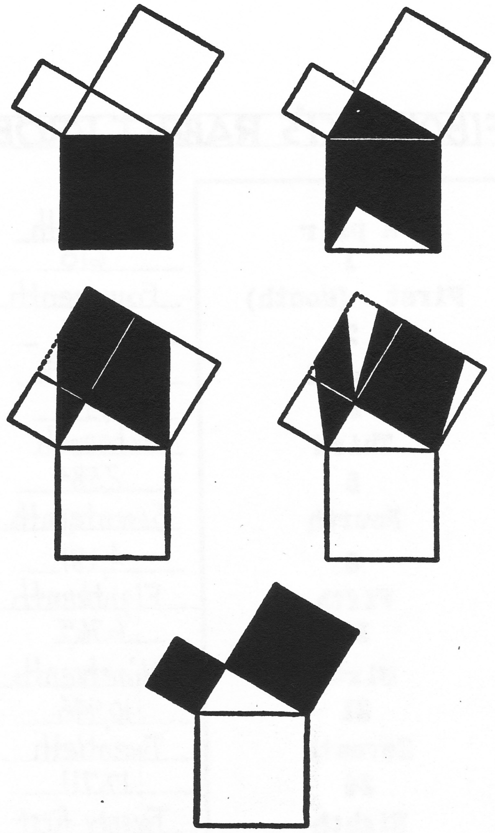
DIVIDE SMALLER BY LARGER

$$\frac{317{,}811}{514{,}229} = 0.618033989$$

THEN -

DIVIDE LARGER BY SMALLER

$$\frac{514{,}229}{317{,}811} = 1.618033989$$



BEHOLD!

Fig. 2.—A Dynamic Proof in a Succession of Five Steps. In the first step (top left) the square of the hypotenuse is marked in black. In the second step (top right) a triangle is added on top of the black square and then an equal triangle is omitted at its base. Between the second and the third stage (middle left) the form of the black area does not change; it is just moved upward until its base line is lifted to the height of its top. In the fourth step (middle right) the black area is split. Each of the two parts takes on the form of a parallelogram and is moved sideways with its bases and heights left unchanged; therefore it preserves its area. This motion can be continued until the black areas reach their last phase and become identical with the squares of the legs. This completes the derivation of the theorem of Pythagoras. It not only demonstrates that the area of the square on the hypotenuse equals the sum of the areas of the squares of the legs, but it shows the actual transformation.

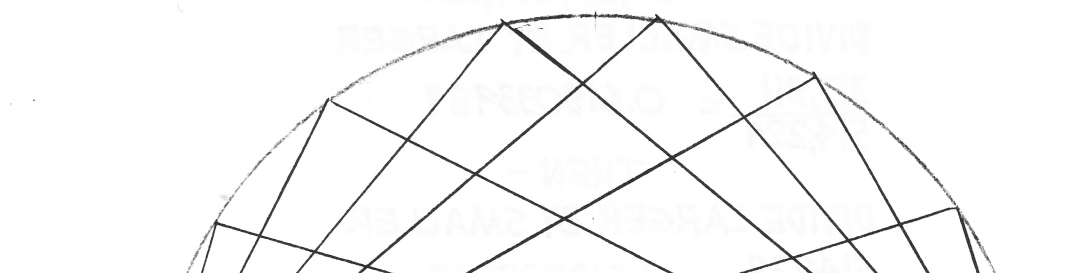
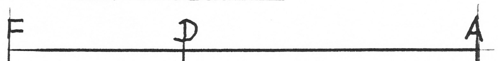


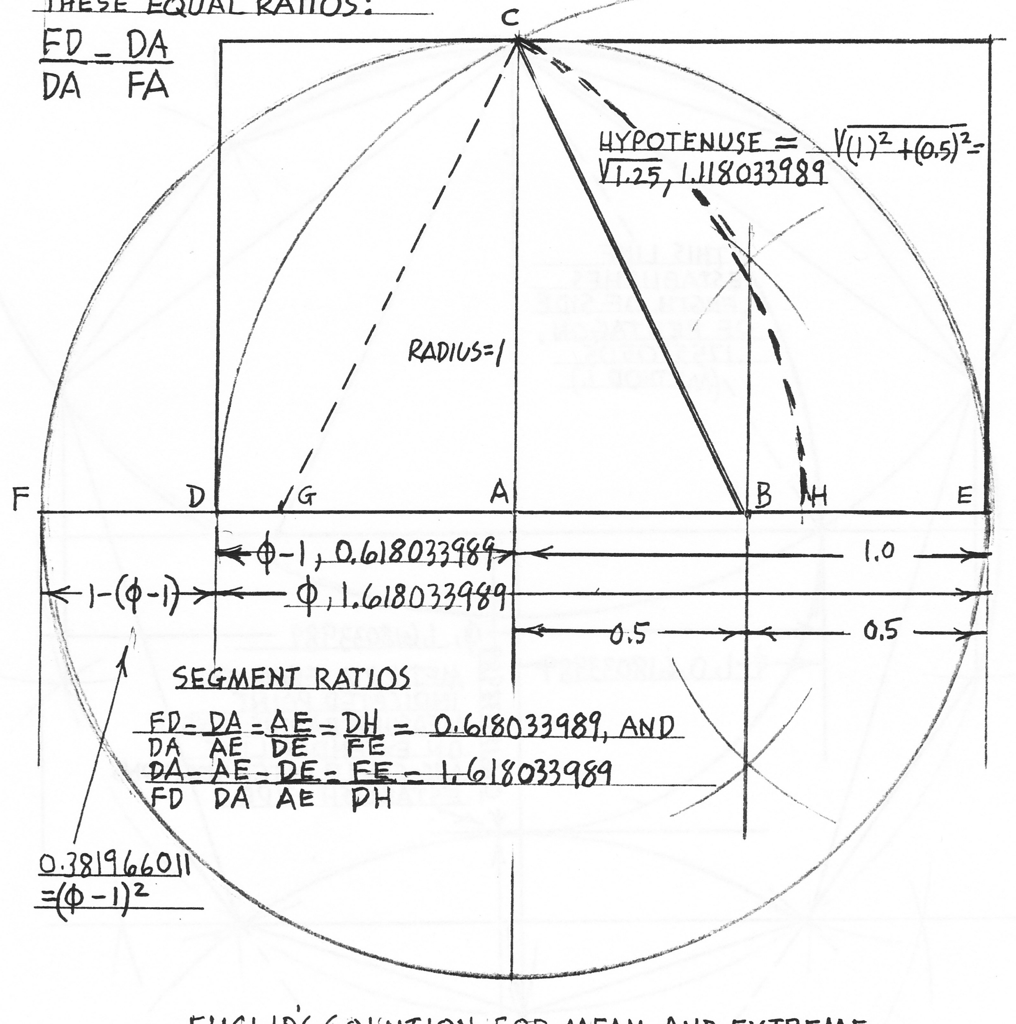
FIG. 3. THALE'S THEOREM

PROBLEM:

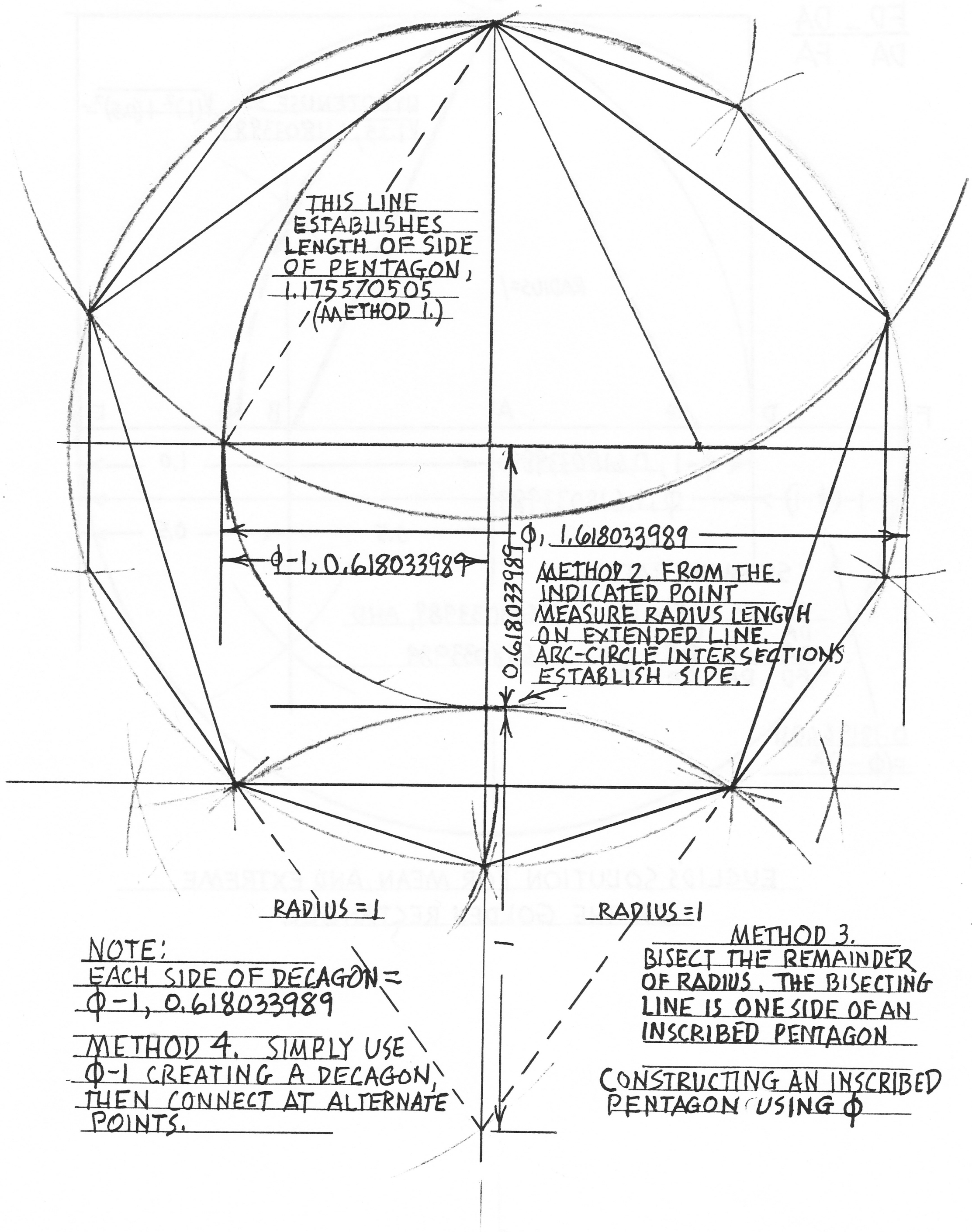


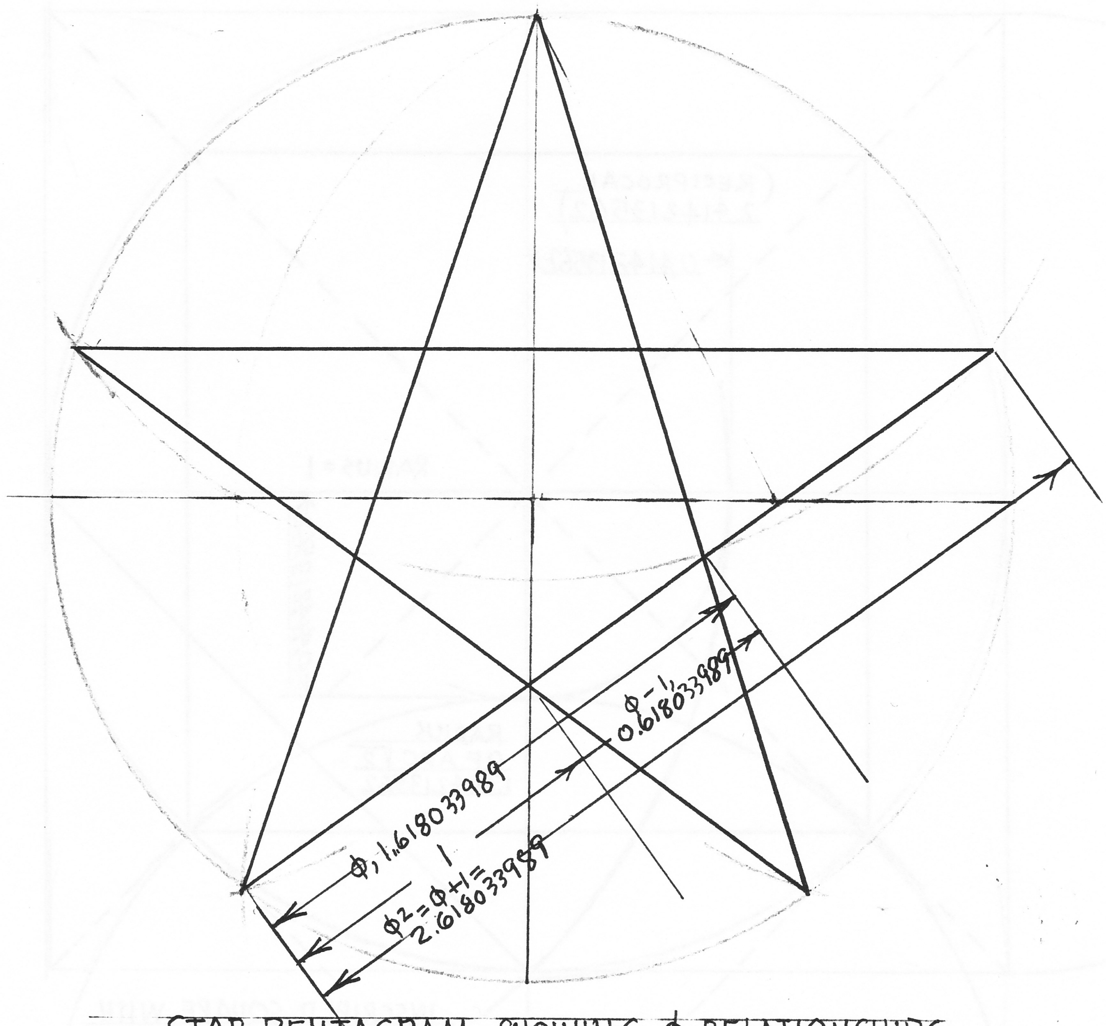
DIVIDE LINE FA INTO SEGMENTS AT A POINT TO CREATE THESE EQUAL RATIOS:

$$\frac{FD}{DA} = \frac{DA}{FA}$$

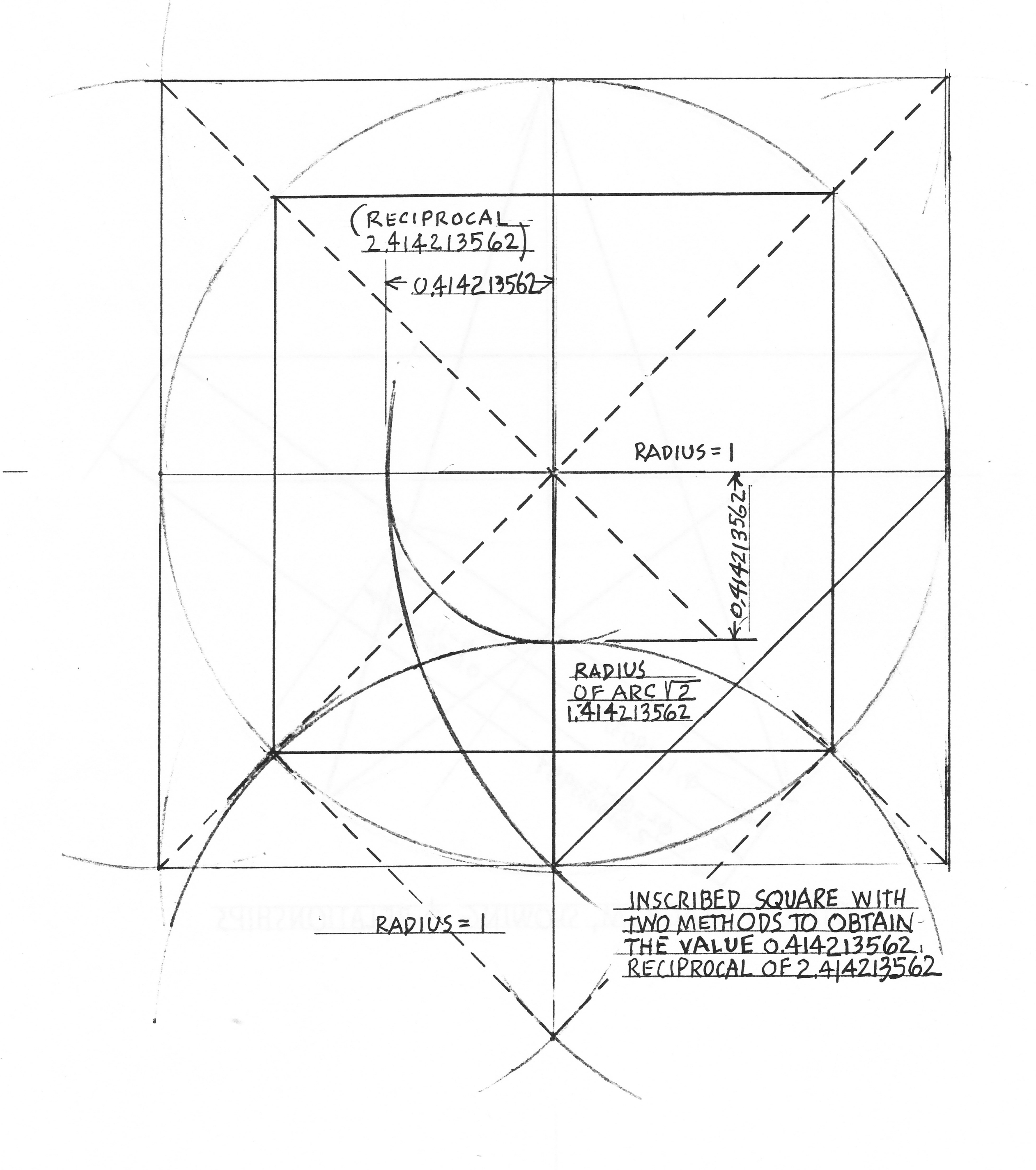


EUCLID'S SOLUTION FOR MEAN AND EXTREME OR THE GOLDEN RECTANGLE

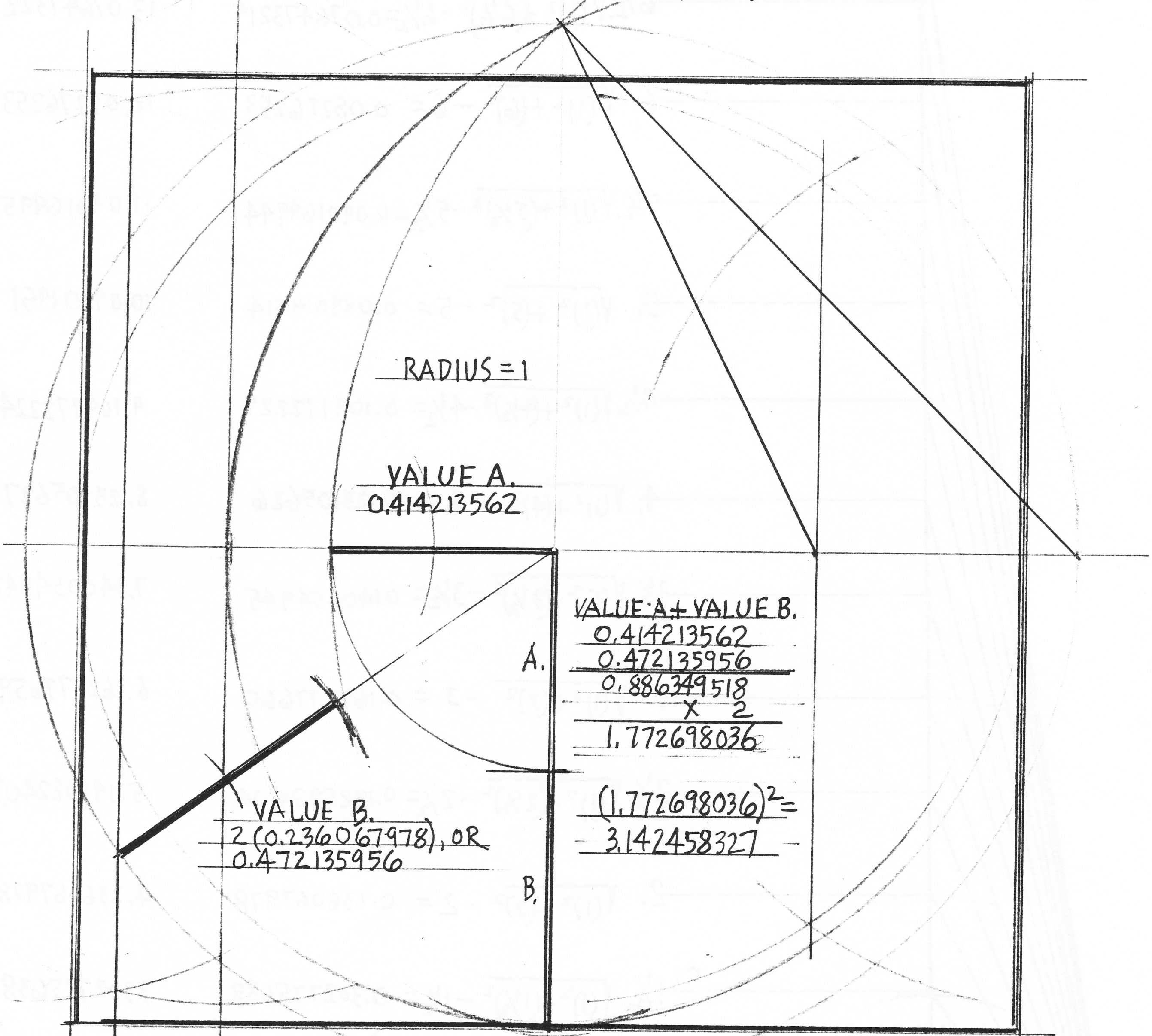

THIS LINE ESTABLISHES LENGTH OF SIDE OF PENTAGON, 1.175570505 (METHOD 1.)
φ, 1.618033989
φ-1, 0.618033989
0.618033989
METHOD 2. FROM THE INDICATED POINT MEASURE RADIUS LENGTH ON EXTENDED LINE. ARC-CIRCLE INTERSECTIONS ESTABLISH SIDE.
RADIUS = 1
RADIUS = 1
1
NOTE: EACH SIDE OF DECAGON = φ-1, 0.618033989
METHOD 4. SIMPLY USE φ-1 CREATING A DECAGON, THEN CONNECT AT ALTERNATE POINTS.
METHOD 3. BISECT THE REMAINDER OF RADIUS. THE BISECTING LINE IS ONE SIDE OF AN INSCRIBED PENTAGON
CONSTRUCTING AN INSCRIBED PENTAGON USING φ



STAR PENTAGRAM, SHOWING φ RELATIONSHIPS


(RECIPROCAL 2.414213562)
0.414213562
RADIUS = 1
0.414213562
RADIUS OF ARC √2 1.414213562
RADIUS = 1
INSCRIBED SQUARE WITH TWO METHODS TO OBTAIN THE VALUE 0.414213562, RECIPROCAL OF 2.414213562

SQUARING THE CIRCLE WITH COMPASS AND STRAIGHT EDGE



EXPRESSION FOR SOLUTION

MODIFICATION BY SELECTIVE BISECTION

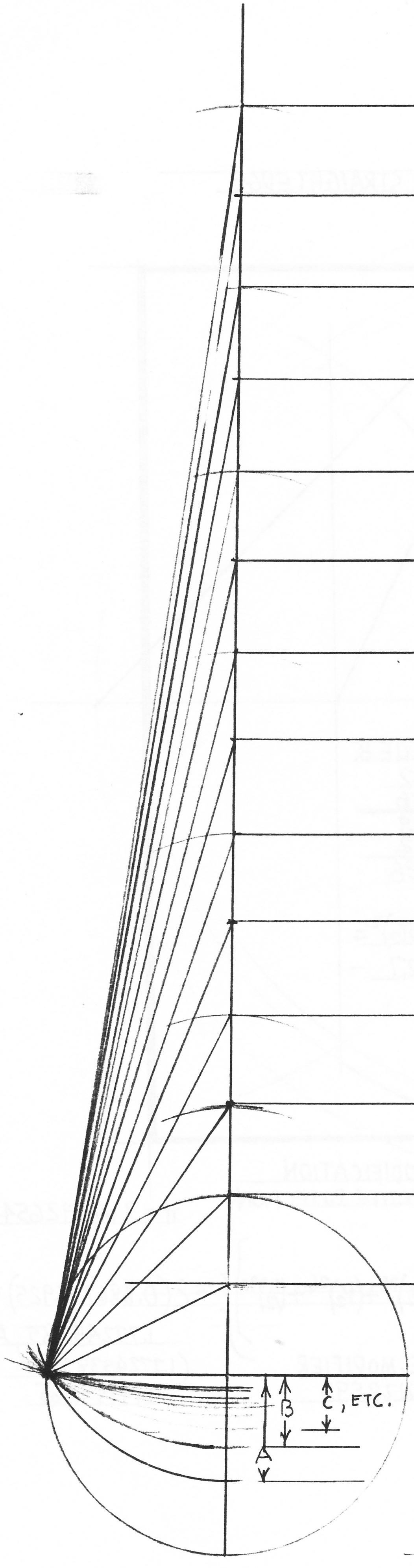
$\pi = 3.141592654$

$$2\left\{2\left[\frac{\dfrac{\left[\sqrt{(1)^2+(\frac{1}{2})^2}-\frac{1}{2}\right]^2}{2}}{\dfrac{\left[\sqrt{(1)^2+(\frac{1}{2})^2}-\frac{1}{2}\right]^2+\left[\sqrt{(1)^2+(\frac{1}{2})^2}-\frac{1}{2}\right]}{2}}\right]+\left[1-2\left(1-\frac{\sqrt{2}}{2}\right)\right]-\left[\left(\tfrac{1}{2}\right)^{13}+\left(\tfrac{1}{2}\right)^{21}+\left(\tfrac{1}{2}\right)^{25}+\left(\tfrac{1}{2}\right)^{26}+\left(\tfrac{1}{2}\right)^{30}\right]\right\}=2(0.886226925)=$$

1.77245385, AND

$(1.77245385)^2 =$ 3.14159265

VALUE B. 2(0.236067978) + VALUE A. 0.414213562 − VALUE, MODIFIER 0.0001122593



POSITION	EXPRESSION	VALUE, CENTER TO ARC INTERSECTION	RECIPROCAL
7.	$\sqrt{(1)^2+(7)^2}-7=$	0.071067812	14.07106781
$6\frac{1}{2}$.	$\sqrt{(1)^2+(6\frac{1}{2})^2}-6\frac{1}{2}=$	0.076473219	13.07647322
6.	$\sqrt{(1)^2+(6)^2}-6=$	0.08276253	12.08276253
$5\frac{1}{2}$.	$\sqrt{(1)^2+(5\frac{1}{2})^2}-5\frac{1}{2}=$	0.090169944	11.09016995
5.	$\sqrt{(1)^2+(5)^2}-5=$	0.099019514	10.09901951
$4\frac{1}{2}$.	$\sqrt{(1)^2+(4\frac{1}{2})^2}-4\frac{1}{2}=$	0.109772229	9.109772224
4.	$\sqrt{(1)^2+(4)^2}-4=$	0.123105626	8.123105627
$3\frac{1}{2}$.	$\sqrt{(1)^2+(3\frac{1}{2})^2}-3\frac{1}{2}=$	0.140054945	7.140054942
3.	$\sqrt{(1)^2+(3)^2}-3=$	0.162277660	6.162277659
$2\frac{1}{2}$.	$\sqrt{(1)^2+(2\frac{1}{2})^2}-2\frac{1}{2}=$	0.192582404	5.192582403
2.	$\sqrt{(1)^2+(2)^2}-2=$	0.236067978	4.236067978
C. $1\frac{1}{2}$.	$\sqrt{(1)^2+(1\frac{1}{2})^2}-1\frac{1}{2}=$	0.302775638	3.302775638
B. 1.	$\sqrt{(1)^2+(1)^2}-1=$	0.414213562	2.414213562
A. $\frac{1}{2}$.	$\sqrt{(1)^2+(\frac{1}{2})^2}-\frac{1}{2}=$	0.618033989	1.618033989

IT IS ASSUMED THAT, WITH ACCURATE COMPUTING, DECIMAL FRACTIONS FOR RECIPROCALS WOULD AGREE EXACTLY, WITH THE WHOLE NUMBERS CONTINUING IN UNIT PROGRESSION LEFT OF THE DECIMAL POINT TO INFINITY, AND THE DECIMAL FRACTIONS CONTINUOUSLY REDUCING TOWARD 0.

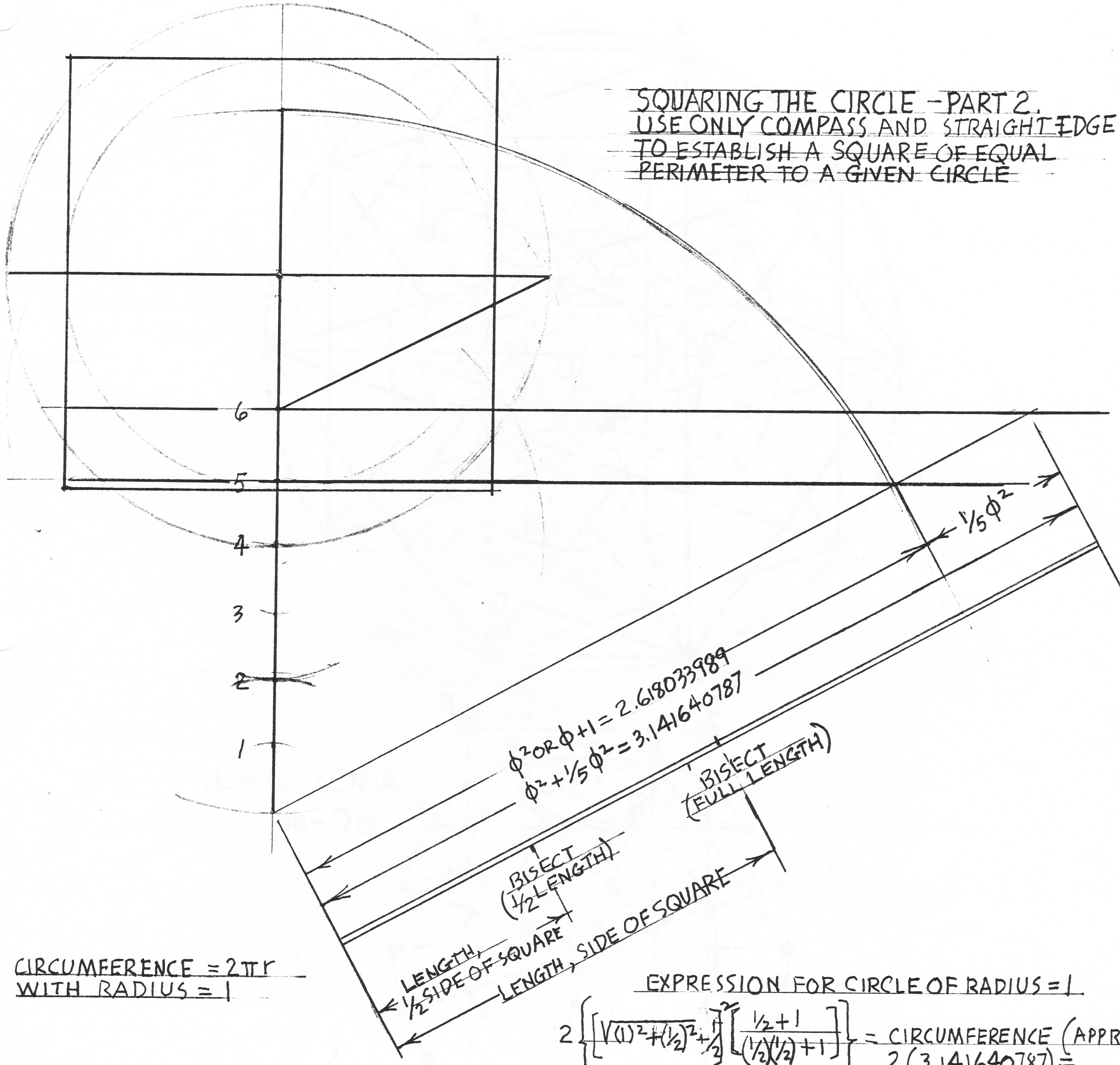
MATHEMATICAL EXPRESSION IS:

$$\sqrt{(1)^2+[(x)(\tfrac{1}{2})]^2}-(x)(\tfrac{1}{2})$$

PROGRAM FOR PERFECT RECIPROCALS

SQUARING THE CIRCLE – PART 2.

USE ONLY COMPASS AND STRAIGHT EDGE TO ESTABLISH A SQUARE OF EQUAL PERIMETER TO A GIVEN CIRCLE



CIRCUMFERENCE = $2\pi r$
WITH RADIUS = 1

EXPRESSION FOR CIRCLE OF RADIUS = 1

$$2\left\{\left[\sqrt{(1)^2+(\tfrac{1}{2})^2}+\tfrac{1}{2}\right]^2\left[\frac{\tfrac{1}{2}+1}{(\tfrac{1}{2})(\tfrac{1}{2})+1}\right]\right\} = \text{CIRCUMFERENCE (APPROX.)}$$

$2(3.141640787) = 6.283281574$

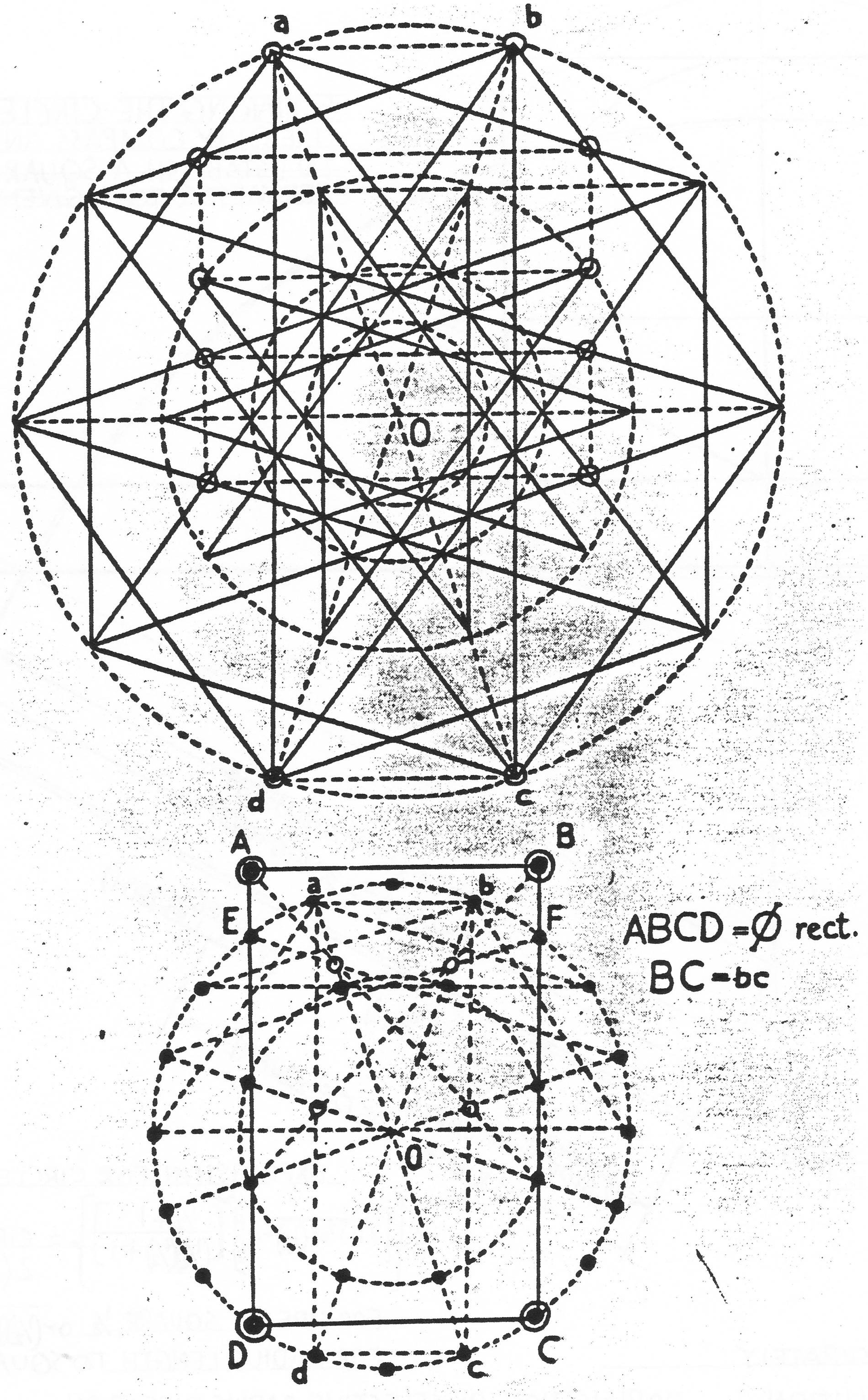
FOR SIDE OF SQUARE, $\frac{1}{4}$ or $(\frac{1}{2})(\frac{1}{2})$

FOR RADIUS LENGTH TO SQUARE $\frac{1}{8}$, OR $(\frac{1}{2})(\frac{1}{2})(\frac{1}{2})$

MORE ACCURATELY

DIAGRAM SOLUTION — MODIFICATION BY SELECTIVE RADIUS BISECTION

$$2\left\{\left[\sqrt{(1)^2+(\tfrac{1}{2})^2}+\tfrac{1}{2}\right]^2\left[\frac{(\tfrac{1}{2})+1}{(\tfrac{1}{2})(\tfrac{1}{2})+1}\right] - \left[(\tfrac{1}{2})^{15}+(\tfrac{1}{2})^{16}+(\tfrac{1}{2})^{19}+(\tfrac{1}{2})^{22}+(\tfrac{1}{2})^{24}+(\tfrac{1}{2})^{25}+(\tfrac{1}{2})^{26}+(\tfrac{1}{2})^{27}\right]\right\} = 2(3.141592654) = 6.283185307$$



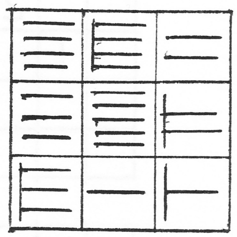
The Gothic Master Diagram

This has been called "the Gothic master diagram" since it is found in one or another form in much Gothic architecture. You will note the appearances of the "golden section ratio."

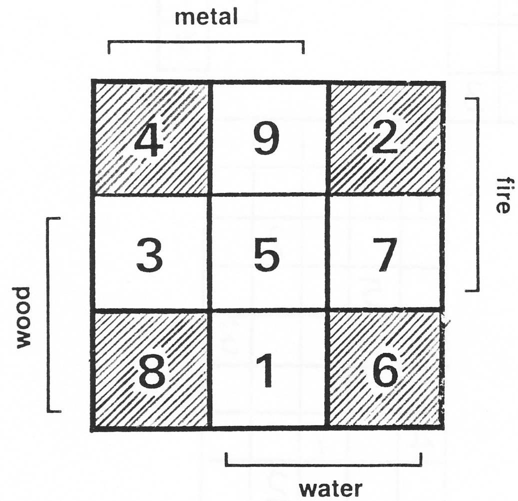
MAGIC SQUARES
Lo Shu

4	9	2
3	5	7
8	1	6

Arabic numerals



Chinese numerals



Yin - female-passive

Yang - male-active

In the series 1, 2, 3 .... 8, 9, there are only eight groupings of 3 which total 15. These are:

9, 5, 1*
9, 4, 2
8, 6, 1
8, 5, 2*
8, 4, 3
7, 6, 2
7, 5, 3*
6, 5, 4*

* Groupings containing 5. There are four such groupings, and no other number appears in more than three. Consequently, 5 must serve as the center. With 15 the required sum of all rows, columns, and of each of the two diagonals, only two combinations of odd-even are permitted, as illustrated:

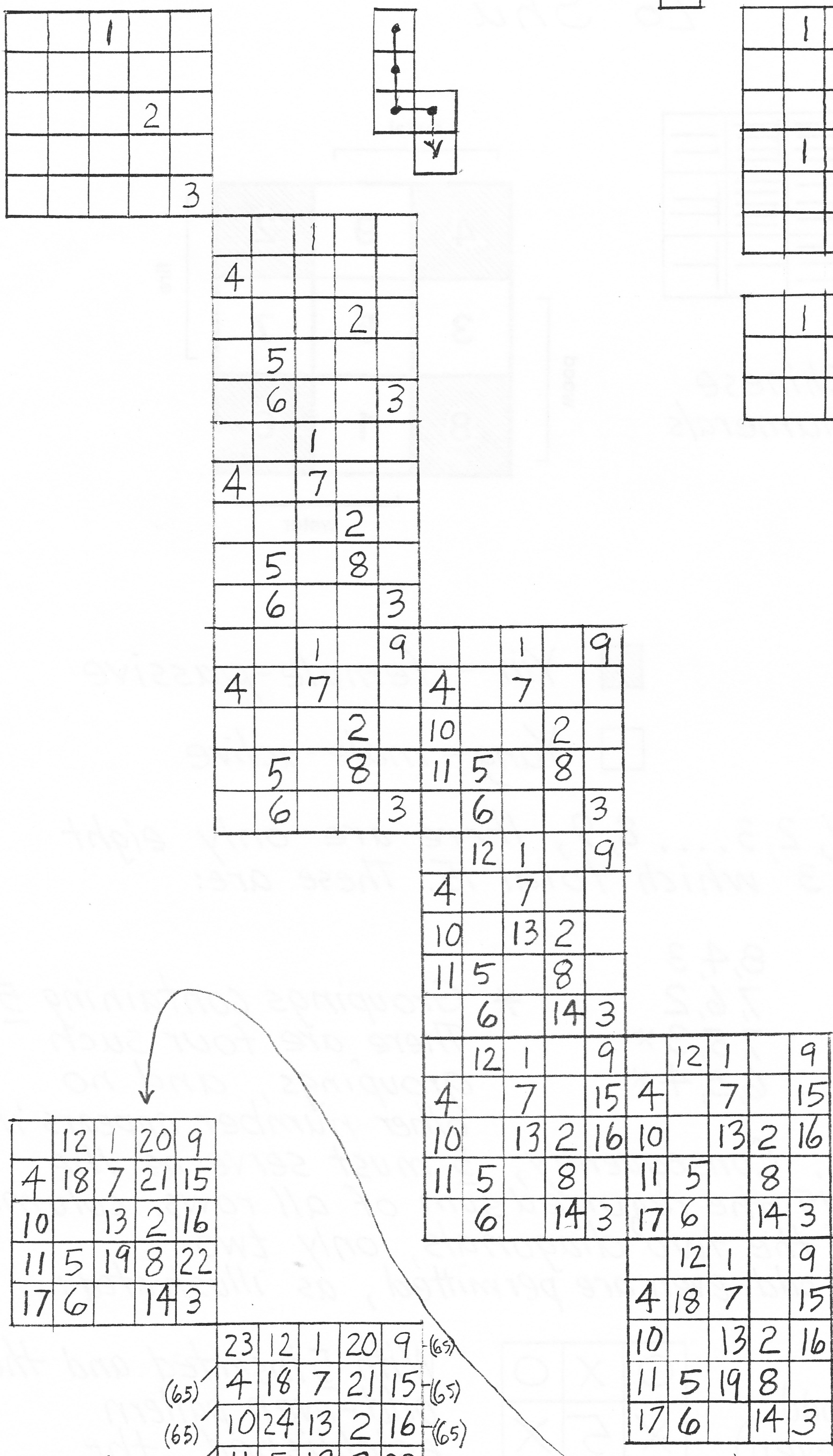
O	X	O
X	X	X
O	X	O

(X, odd
O, even)

O	X	O
X	5	X
O	X	O

With 5 inserted and the odd-even pattern established, the solution is easy.

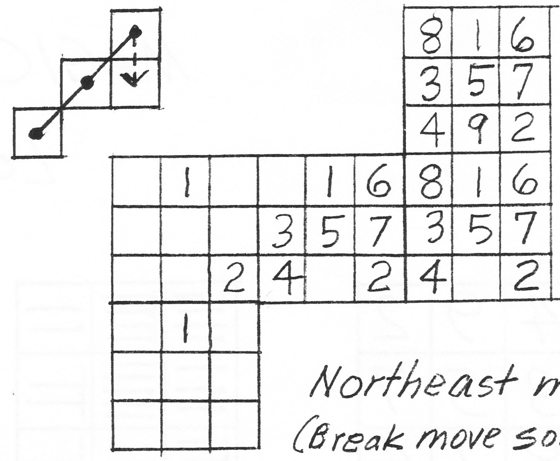
Knight's Move Method-Magic-5 (Break move south)



Result

23	12	1	20	9	(65)
4	18	7	21	15	(65)
10	24	13	2	16	(65)
11	5	19	8	22	(65)
17	6	25	14	3	(65)
(65)	(65)	(65)	(65)	(65)	(65)

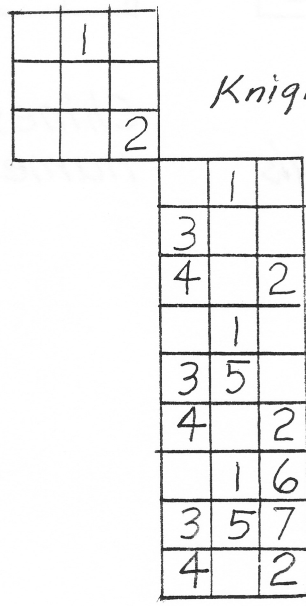
(65)



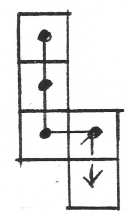
Result

8	1	6
3	5	7
4	9	2

Northeast method, (Break move south)

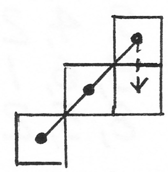


Knight's Move method (Break move south)



Result

8	1	6
3	5	7
4	9	2



Northeast method Magic-5 Break move south

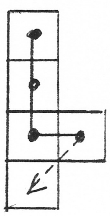
(65)	17	24	1	8	15
(65)	23	5	7	14	16
(65)	4	6	13	20	22
(65)	10	12	19	21	3
(65)	11	18	25	2	9
(65)	(65)	(65)	(65)	(65)	(65)

(65) (65) (65) (65) (65) (65)

Result

Note from the foregoing methods results of ordinary magic plus, for the northeast method produces uniform southeast diagonals totaling 65, while the knight's move produces northeast diagonals totaling 65. This phenomenon is observed only for prime number squares.

A SUPERIOR SQUARE containing ordinary magic, i.e., equal rows, columns, and diagonals, plus all diagonals totaling 65, can be generated by a powerful algorithm using the same knight's move, but with a single space break move to the southwest. Again, this is good only for prime number squares or squares which are products of prime numbers 5 and higher.



	(65)	(65)	(65)	(65)	(65)	(65)
(65)	9	12	20	23	1	(65)
(65)	18	21	4	7	15	(65)
(65)	2	10	13	16	24	(65)
(65)	11	19	22	5	8	(65)
(65)	25	3	6	14	17	(65)
	(65)	(65)	(65)	(65)	(65)	

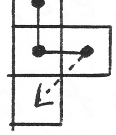
Notice that all numbers opposite and equidistant from the center total 26, or double the center amount.

9	12	20	23	1	9	12	20	23	1
18	21	4	7	15	18	21	4	7	15
2	10	13	16	24	2	10	13	16	24
11	19	22	5	8	11	19	22	5	8
25	3	6	14	17	25	3	6	14	17
9	12	20	23	1	9	12	20	23	1
18	21	4	7	15	18	21	4	7	15
2	10	13	16	24	2	10	13	16	24
11	19	22	5	8	11	19	22	5	8
25	3	6	14	17	25	3	6	14	17

Superiority of this omnidirectional perfection is demonstrated above with 4 squares placed together, as any enclosed square of 5 sides is also perfect.

Additional Example

	(175)	(175)	(175)	(175)	(175)	(175)	(175)	(175)
(175)	31	12	42	16	46	27	1	(175)
(175)	20	43	24	5	35	9	39	(175)
(175)	2	32	13	36	17	47	28	(175)
(175)	40	21	44	25	6	29	10	(175)
(175)	22	3	33	14	37	18	48	(175)
(175)	11	41	15	45	26	7	30	(175)
(175)	49	23	4	34	8	38	19	
	(175)	(175)	(175)	(175)	(175)	(175)	(175)	



This algorithm is equally effective in its rotated positions, or used in its mirror image form.

EVEN MAGIC SQUARES, OF INDIAN ORIGIN

With all rows, columns, and <u>all</u> diagonals equal (i.e., all 34)

7	12	1	14
2	13	8	11
16	3	10	5
9	6	15	4

Dudhai (or Jaina)

15	10	3	6
4	5	16	9
14	11	2	7
1	8	13	12

Gwalior

These are super-magic-4s, and either may be used to build super-magic 16, or larger, squares, as illustrated below.

	(2056)	(2056)	(2056)	(2056)	(2056)	(2056)	(2056)	(2056)	(2056)	(2056)	(2056)	(2056)	(2056)	(2056)	(2056)	(2056)
(2056)	103	108	97	110	183	188	177	190	7	12	1	14	215	220	209	222
(2056)	98	109	104	107	178	189	184	187	2	13	8	11	210	221	216	219
(2056)	112	99	106	101	192	179	186	181	16	3	10	5	224	211	218	213
(2056)	105	102	111	100	185	182	191	180	9	6	15	4	217	214	223	212
(2056)	23	28	17	30	199	204	193	206	119	124	113	126	167	172	161	174
(2056)	18	29	24	27	194	205	200	203	114	125	120	123	162	173	168	171
(2056)	32	19	26	21	208	195	202	197	128	115	122	117	176	163	170	165
(2056)	25	22	31	20	201	198	207	196	121	118	127	116	169	166	175	164
(2056)	247	252	241	254	39	44	33	46	151	156	145	158	71	76	65	78
(2056)	242	253	248	251	34	45	40	43	146	157	152	155	66	77	72	75
(2056)	256	243	250	245	48	35	42	37	160	147	154	149	80	67	74	69
(2056)	249	246	255	244	41	38	47	36	153	150	159	148	73	70	79	68
(2056)	135	140	129	142	87	92	81	94	231	236	225	238	55	60	49	62
(2056)	130	141	136	139	82	93	88	91	226	237	232	235	50	61	56	59
(2056)	144	131	138	133	96	83	90	85	240	227	234	229	64	51	58	53
(2056)	137	134	143	132	89	86	95	84	233	230	239	228	57	54	63	52

(2056) (2056) (2056) (2056) (2056) (2056) (2056) (2056) (2056) (2056) (2056) (2056) (2056) (2056) (2056) (2056) [diagonals, right side]

(2056) (2056) (2056) (2056) (2056) (2056) (2056) (2056) (2056) (2056) (2056) (2056) (2056) (2056) (2056) (2056) [diagonals, bottom]

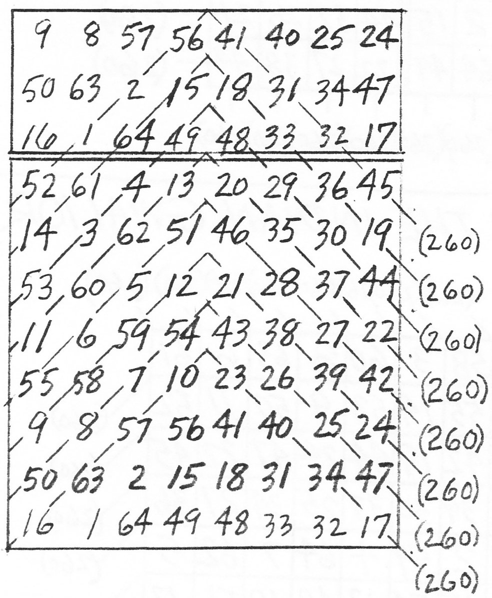
The overall pattern for the above super-magic 16 is the Dudhai magic-4. In the 1 position is the Dudhai magic 4. In the 2 position there is a square containing the numbers 17 through 32 inserted in Dudhai sequence. In the 3 position are numbers 33 through 48 in the same Dudhai sequence, etc. The result is a super-magic 16.

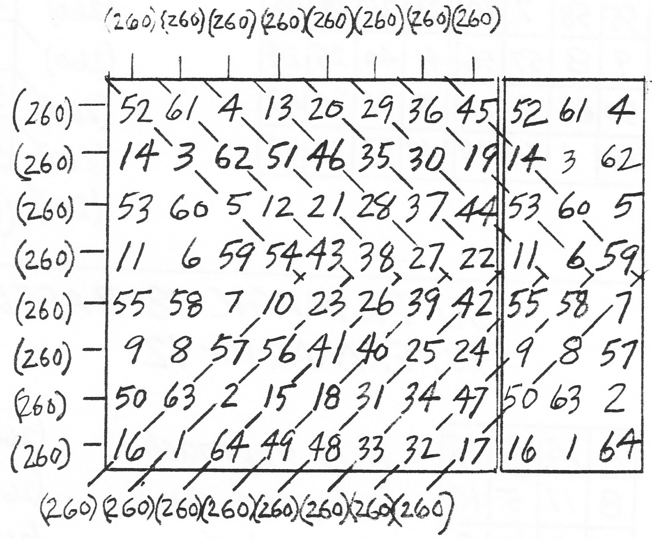
An alternate solution is to create a series for the 1 position. Use 1, 17, 33, 49 ... 225, 241 for this block, inserting in Dudhai sequence. In 2 position use 2, 18, 34, 50 ... 226, 242, etc.

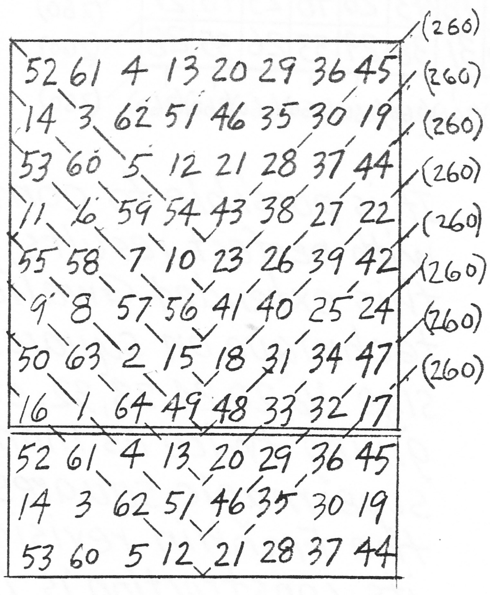
FRANKLIN'S MAGIC-8

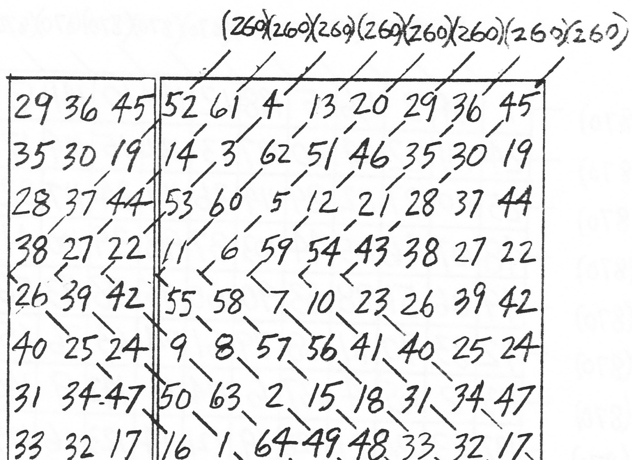
52	61	4	13	20	29	36	45
14	3	62	51	46	35	30	19
53	60	5	12	21	28	37	44
11	6	59	54	43	38	27	22
55	58	7	10	23	26	39	42
9	8	57	56	41	40	25	24
50	63	2	15	18	31	34	47
16	1	64	49	48	33	32	17

Franklin described his Magic-8, of which he gave a copy to his friend, "the learned Mr. Logan" in a letter, "every straight row, horizontal or vertical, of 8 numbers added together makes 260, and half each row 130." Bent rows are as illustrated. "And, lastly the 4 corner numbers, with the 4 middle numbers make 260." Each 4 number cluster totals 130.









In the quest for super-magic squares, one can rework one side of the Franklin-8, reverse the direction of the diagonals, and create a perfect magic square, as follows:

52	61	4	13	20	29	36	45
14	3	62	51	46	35	30	19
53	60	5	12	21	28	37	44
11	6	59	54	43	38	27	22
55	58	7	10	23	26	39	42
9	8	57	56	41	40	25	24
50	63	2	15	18	31	34	47
16	1	64	49	48	33	32	17

52	61	4	13	42	39	26	23
14	3	62	51	24	25	40	41
53	60	5	12	47	34	31	18
11	6	59	54	17	32	33	48
55	58	7	10	45	36	29	20
9	8	57	56	19	30	35	46
50	63	2	15	44	37	28	21
16	1	64	49	22	27	38	43

(Every row, column and diagonal marked (260).)

SUPER MAGIC-8 BASED ON THE INDIAN GWALIOR, ALSO SUPERMAGIC-12

1	14	4	15
8	11	5	10
13	2	16	3
12	7	9	6

This approach is adapted and expanded → for 8 square, and 12 square ↓

1	58	3	60	8	63	6	61
16	55	14	53	9	50	11	52
17	42	19	44	24	47	22	45
32	39	30	37	25	34	27	36
57	2	59	4	64	7	62	5
56	15	54	13	49	10	51	12
41	18	43	20	48	23	46	21
40	31	38	29	33	26	35	28

(Every row, column and diagonal marked (260).)

1	134	3	136	5	138	12	143	10	141	8	139
24	131	22	129	20	127	13	122	15	124	17	126
25	110	27	112	29	114	36	119	34	117	32	115
48	107	46	105	44	103	37	98	39	100	41	102
49	86	51	88	53	90	60	95	58	93	56	91
72	83	70	81	68	79	61	74	63	76	65	78
133	2	135	4	137	6	144	11	142	9	140	7
132	23	130	21	128	19	121	14	123	16	125	18
109	26	111	28	113	30	120	35	118	33	116	31
108	47	106	45	104	43	97	38	99	40	101	42
85	50	87	52	89	54	96	59	94	57	92	55
84	71	82	69	80	67	73	62	75	64	77	66

(Every row, column and diagonal marked (870).)

It is possible to continue, with perfect results, using this extended Gwalior technique on squares of size 16, 20, 24, 28, 32, etc., this giving twice as many supermagic squares as the Franklin revisions, for its construction is limited to 8, 16, 24, 32, 40, etc.

FRANKLIN MAGIC-16

200	217	232	249	8	25	40	57	72	89	104	121	136	153	168	185
58	39	26	7	250	231	218	199	186	167	154	135	122	103	90	71
198	219	230	251	6	27	38	59	70	91	102	123	134	155	166	187
60	37	28	5	252	229	220	197	188	165	156	133	124	101	92	69
201	216	233	248	9	24	41	56	73	88	105	120	137	152	169	184
55	42	23	10	247	234	215	202	183	170	151	138	119	106	87	74
203	214	235	246	11	22	43	54	75	86	107	118	139	150	171	182
53	44	21	12	245	236	213	204	181	172	149	140	117	108	85	76
205	212	237	244	13	20	45	52	77	84	109	116	141	148	173	180
51	46	19	14	243	238	211	206	179	174	147	142	115	110	83	78
207	210	239	242	15	18	47	50	79	82	111	114	143	146	175	178
49	48	17	16	241	240	209	208	177	176	145	144	113	112	81	80
196	221	228	253	4	29	36	61	68	93	100	125	132	157	164	189
62	35	30	3	254	227	222	195	190	163	158	131	126	99	94	67
194	223	226	255	2	31	34	63	66	95	98	127	130	159	162	191
64	33	32	1	256	225	224	193	192	161	160	129	128	97	96	65

B. Franklin inv. I. Ferguson delin. J. Mynde sc.

All 4-squares contained total 2056.

All bent diagonals total 2056.

All half rows and half columns total 1028.

A magic square of squares. Much as he might admonish his son to be unswervingly attentive as clerk of the Pennsylvania Assembly, Franklin himself had often found the debates "so unentertaining that I was induc'd to amuse myself with making magic Squares or Circles, or any thing to avoid Weariness." Later these squares and circles were considered worthy of publication in his scientific works. Printed in James Ferguson, *Tables and Tracts* (London, 1767). Yale University, Franklin Collection.

FLEXING

9	12	20	23	1
18	21	4	7	15
2	10	13	16	24
11	19	22	5	8
25	3	6	14	17

1. Position 1 contains 9. Place 1 in 9th square.
2. Position 2 contains 12. Place 2 in 12th square.
3. Position 3 contains 20. Place 3 in 20th square.
4. Position 4 contains 23. Place 4 in 23rd square.
5. Position 5 contains 1. Place 5 in 1st square.
6. Continue second row, etc. Completed square as shown

↓

5				
			1	
	2			
				3
		4		

→

5	11	22	8	19
23	9	20	1	12
16	2	13	24	10
14	25	6	17	3
7	18	4	15	21

Both the original and the derived squares are super-magic.