Curlis Brasher
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The Golden Age - or Was It?

A few weeks ago my daughter asked me to help her study for a science test by looking through the assigned chapters and asking questions about the subjects presented. As I leafed through the book I came upon a discussion of Avogadro's number. Having heard the term but not knowing what it meant I read the discussion with considerable interest. It seems that Avogadro's number represents the number of atoms present in that amount of an element equal to its atomic weight measured in grams. As I understand it, if the atomic weight of an element is 20, than in 20 grams of the element will be Avogadro's number of atoms. This number is 6×10^{23}, or 6 followed by twenty-three zeros. Avogadro's number is named for the scientist who first suggested it and is very important in the study of chemistry because chemists can use it to predict the amount of elements and compounds that will combine in chemical reactions.

This chance encounter with Avogadro's number led me to think of other men whose names are linked with numbers. There is Leonardo of Pisa, the medieval mathematician who is better known as Fibonacci, whose name is given to the numerical sequence which he discovered. The Fibonacci sequence is produced by starting with 1 and adding the last two numbers to arrive at the next: 1,1,2,3,5,8,13,21,34,.... This sequence is interesting enough in itself, but it becomes quite remarkable when we realize that the ratios of adjacent Fibonacci numbers appear everywhere in nature. The spiral shell of a chambered nautilus, and the spiraled florets of a daisy are arranged in the ratio of 21:34. The scales of a pine cone are arranged in a 5;8 ratio; and the bumps on a pineapple are arranged in an 8;13 ratio. Fibonacci ratios have been found in the curve of an elephant's tusk, the horns of a wild sheep, and even in canaries' claws.

Consider also that the ratio between any two adjacent Fibonacci numbers greater than 3 is approximately 1 to 1.6. This is the so-called "Golden Ratio" which is involved in art, architecture, and esthetics. The "Golden Rectangle" is the figure whose sides bear this relationship to each other and which is said to be the most visually satisfying of all geometric forms. The Parthenon and many other classical buildings were built on the principle of the Golden Rectangle and even today architects make use of it.

Observation reveals many correlations between the world of nature and the world of mathematics. What connection is there between the man-made abstract world of numbers and the curve of a seashell? Is it coincidence or is there really a connection? Throughout recorded history men have tried to explain these things by scientific answers and by metaphysical means. There have always been those who beleive that numbers are not mere abstractions but are living forces. We label these believers numerologists. They are seldom mathematicians but they can invoke the name of a great mathematician who was also a believer in the mystic power of numbers - Pythagoras. Let us look, as laymen, at Pythagoras and his school and try to place their work in the context of mathematics history. Their time has been labeled the Golden Age of Mathematics, but was it, really? In any event it was the beginning of a great era of man's intellectual history.

Pythagoras is a well known name and one that was feared by high school students in my school days because our plane geometry course was crowned with the proof of the famous theorem about the squares on the sides of a right triangle. What is less well known however is the fact that most of the work of Pythagoras and his school was done in number theory; but I hope to show that their work in this field delayed the growth of mathematics as much as it advanced it.

Pythagoras was born on the Aegean island of Samos around 572 B.C. He is supposed to have studied under Thales of Miletus, who is considered to be the father of demonstrative geometry. It won't hurt to digress a moment to think about Thales. He was a many-sided genius who like John Maynard Keynes made a fortune in trade early in life and retired to devote his time to mathematics, astronomy, and philosophy. He was also a statesman and an engineer. He was considered one of the Seven Wise Men of classical times. Thales lived for a while in Egypt where he must have become acquainted with Egyptian mathematics and geometry. He is also thought to have traveled in Babylonia where he studied astronomy. At that time the Babylonians were far advanced over Greece and Egypt in astronomy and number theory. Since around 3000 B.C. the Babylonians and their predecessors had been using a number system based on 60 instead of 10 and had developed an advanced numeration system and had made many difficult calculations in astronomy. Thales was the most qualified tacher of mathemtics of his time and he must have had an apt pupil in Pythagoras.

Pythagoras is thought to have gone to Egypt, Assyria, and Babylon after studying with Thales and some scholars claim that he even went to India. The next thing that is known for certain is that he founded a school at Crotona, a Greek colony on the instep of the Italian boot. His pupils became known as the Pythagorean Brotherhood. They were sworn to secrecy, lived a communal life as a semi-religious body, and cultivated mystical beliefs in the power of numbers. The members wore a five-pointed star as a badge. The course of study included arithmetic, geometry, astronomy, and music. Arithmetic did not mean mere adding and subtracting but rather what we call number theory. The Greeks used the term logistic to describe the studies that we call arithmetic today.

It is impossible to know how many of their discoveries should be attributed to Pythagoras personally because all of the works of the school were published in his name. The members of the brotherhood gave Pythagoras all the credit and even refused to divulge their own names.

In fact the methods and doctrines of the Pythagoreans became so mysterious and secretive that theycaused people to be afraid of them. The brothers were shunned and even threatened. They also had the misfortune to be treated as pawns by the rulers of bickering Greek city-states who tried to enlist their mystical aid in fighting their wars.

Pythagoras is said to have died in 501 B.C. Probably he was murdered. At any rate his school was destroyed at about the same time and the brothers were either killed or forced to flee. Those who survived eventually reassembled and the school of Pythagoras continued for about two hundred more years.

Pythagoras did the first experimental work on the mathematical foundation of music. He learned that by doubling the length of a string the note sounded when the string is plucked is exactly one octave lower. He found that the entire scale could be played simply by lengthening the string according to simple increments expressible as the ratios of whole numbers. From this work with music Pythagoras was convinced that all harmony, all beauty, all nature could be expressed by whole number relationships. He thought he had discovered the guiding principle of the universe. Even the planets movedin harmony to the "music of the spheres".

The Pythagoreans were so satisfied with their whole number universe that their work began to tend more and more to numerology and less and less to mathematics. They said that numbers were the Cause of all things, the generating force which created all nature and that all nature could be understood if numbers were understood. The Pythagoreans began to classify numbers and they apparently were the first to catalog prime numbers. They assigned meanings to numbers: All the odd numbers were masculine except the number one. It was considered sacred as the generator of all other numbers. The even numbers were feminine. The number five was the symbol for marriage since it is the sum of 2 and 3 the first feminine number and the first masculine number.

The Pythagoreans also discovered "amicable"numbers. Two numbers are said to be amicable if each is the sum of the factors of the other. For example, 220 and 284 are amicable since the factors of 220 are 1, 2, 4, 5, 10, 11, 20, 22, 44, 55, 110 and they add up to 284, while the factors of 284 are 1, 2, 4, 71, and 142 and their sum is 220. Pythagoras himself is said to have discovered this pair of amicable numbers. Numerologists maintain that two talismans bearing these numbers will seal perfect friendship between the wearers. These numbers came to be very important in magic and astrology and the casting of horoscopes.

Curiously enough no new pair of amicable numbers was reported until 1636 when the great French number theorist Pierre de Fermat announced that 17,296 and 18,416 were amicable. Two years later Rene Decartes found a third pair. In the 18th Century the great Swiss mathematician Leonhard Euler produced a list of 60 pairs. In 1886 a 16 year old Italian boy found the overlooked pair, 1184 and 1210. Now more than

400 pairs of amicable numbers are known.

Other numbers having mystical connections essential to numerologists and astrologists were perfect, deficient, and abundant numbers. A number is perfect if it is the sum of its factors, deficient if it is greater than the sum of its factors, and abundant if it is less than the sum of its factors.

In the Middle Ages even Christian churchmen were believers in the mystical power of perfect numbers. According to one monkish writer God's first creation of the world was perfect because it was done in 6 days, 6 being a perfect number. On the other hand the world created after the Deluge is imperfect since all mankind is descended from the eight souls on Noah's ark and 8, being greater than 1 plus, 2, plus 4 is deficient. Until 1952 there were only 12 known perfect numbers, all of them even numbers. The first three perfect numbers are 6, 28, and 496. The last proposition of Euclid's Elements gives a formula for producing a perfect number. All perfect numbers produced by this formula are even numbers and Euler proved that every even perfect number must be of this form, but no one has ever proved the existence or nonexistence of odd perfect numbers. This question, the existence of odd perfect numbers, is one of the celebrated unsolved problems of number theory. I am told that mathematicians are certain that no odd perfect numbers smaller than 2,000,000 exist. In 1952 a computer, programmed to Euclid's formula, found five more perfect numbers. Since that time other computers have brought the list of perfect numbers to 18.

The most famous work of Pythagōas, of course, is the right triangle theorem, namely that the sum of the squares on the legs equalled the square on the hypotenuse. Recently discovered cuneiform tablets indicate that the Babylonians in the time of Hammurabi knew this theorem, however, it is reasonable to believe that Pythagoras gave the first general proof of it.

Closely related to this theorem is the problem of finding integers which can represent the legs and hypotenuse of the right triangle. The numbers, 3,4, and 5 obviously do. In fact the Egyptians had for centuries used a rope with knots tied to represent these intervals as a surveying tool to locate land boundaries after the Nile's annual flood moved the boundary stones. Numbers of this sort are called Pythagorean triples. The Pythagoreans devised a formula to find some triples and it was probably while working on this problem that they made one of the most important discoveries in the history of mathematics. This was the discovery of irrational numbers and it shook the Pythagorean brotherhood to its very foundations.

To see how important this discovery was it might be well to review the way our number system is constructed. We begin with the idea of counting numbers, 1, 2, 3, ... which are sufficient for enumerating a collection of objects or stating the order of things. From counting numbers we proceed to the idea of whole numbers, which are the counting numbers, plus zero. Next in order of development are the integers, which add negative numbers to the set of whole numbers. Integers would be sufficient for all the needs of society if everything always worked out in exact integral units. We know that they do not, so fractions are required to solve many problems. Therefore we add the set of rational numbers to our number system to include integers and fractions.

The Pythagoreans' work with musical strings, amicable numbers, perfect numbers, and Pythagorean triples had convinced them that the entire number system had been completed with the idea of rational numbers. In their worship of numbers as divine they insisted that everything had to come out even. The only things that apparently did not come out even were attributable to calculation errors rather than to the numbers themselves. No doubt it was the right triangle that finally convinced them that not all numbers are rational, and when they found this out they hid this information from everybody who was not in the Brotherhood. Their basic assumption that everything in nature depended on the whole numbers was completely upset. A crack as large as a chasm had appeared in the foundation of their religion, philosophy, and science. In theological terms, the whole ground of their being was shaken. I do not want to exaggerate the significance of their discovery, but neither do I wish to minimize it because some extremely important results for the future development of mathematics flowed from it.

It is not too far-fetched to compare the feelings of the Pythagoreans to the response of the Pope and the College of Cardinals to some incontrovertible evidence that Jesus never existed.

Some of the brotherhood tried to pretend that thediscovery had never been made; that irrational numbers did not exist. They also triedto keep the secret within the Brotherhood. One story says that a member was drowned at sea for revealing the secret to outsiders.

We may smile today at this mixture of mysticism and science; but we must regret the turn of events which followed. The Pythagoreans turned away from number theory, and such was their influence upon other mathematicians and teachers that these, too, assumed that work on numbers had reached a dead end. Greek mathematics restricted itself almost entirely to shape-geometry. In this way mathematicians could draw and encompass these irrational magnitudes even if they could not measure them. The emphasis of Greek mathematics thus was placed on restricting and limiting the field of mathematics instead of expanding it. Classical mathemtics never recovered its vision. Their work in their limited field was symmetrical, beautiful and rigorous but even Archimedes, one of the three or four greatest mathematicians of all time could not break out of this way of thinking. The result was a stagnation that lasted until the 14th Century A. D. when Fibonacci introduced the zero to the Western world along with Hindu-Arabic numerals.

The Pythagoreans deserve our praise for what they did, nor should they be too much censured. It may be that it is too much to expect that a people evolving from barbarism and nature worship, worshipping order, symmetry, and beauty; fearing disorder and yet experiencing it everywhere in their chaotic city-states, could have found room in their system of values for a number system which seemedto contain the very disorderly, irrational elements they feared so much.

Perhaps we can learn a lesson from them. It is dangerous to try to select the truth we prefer and hide that which we do not understand. Utility as well as honesty dictates that we pursue the truth no matter how uncomfortable it may be to cherished beliefs.